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Two values for transferable utility games with coalition and graph structure[1]

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Abstract

In this paper we introduce and characterize two new values for transferable utility games with graph restricted communication and a priori unions. Both values are obtained by applying the Shapley value to an associated TU-game. The graph-partition restricted TU-game is obtained by taking the Myerson graph restricted game and of that the Kamijo partition restricted game. In this game the dividend of any coalition that is neither a subset of a union nor a union of unions is zero. The partition-graph restricted TU-game is obtained by taking the partition restricted game and of that the graph restricted game. In this game the dividend of any coalition that is not connected in the graph is zero. We apply the values to an economic example in which the players in a union represent the cities in a country and the graph represents a network of natural gas pipelines between the cities.

Keywords: Cooperative games, coalition structures, graphs, Shapley value.

JEL code: C71

1 Introduction

A cooperative game with transferable utility in characteristic function form, or simply a TU-game, is a rudimentary model of cooperation among (economic) agents. TU-games were introduced by von Neumann and Morgenstern (1944) and have since become a central object of study in the field of cooperative game theory. An important objective in this field is the determination of a value for each agent, referred to as player, in a TU-game.[1] Unlike strategic solution concepts, values are usually defined axiomatically. Some desirable properties are stated and it is shown that there exists a unique value that satisfies these properties. For instance, the Shapley value (Shapley, 1953), one of the principal values in cooperative game theory, is the unique value that satisfies the axioms of 'symmetry', 'carrier' and 'additivity'.

Aumann and Drèze (1974) were one of the first to consider restrictions on cooperation possibilities of players in a TU-game by partitioning the set of players in a number of a priori unions (elements of the partition). Nowadays, TU-games with a partition of the set of players are known as TU-games with coalition structure, or TU-games with a priori unions. To obtain a value for a TU-game with coalition structure, Aumann and Drèze (1974) assumed that the players in the game are only allowed to cooperate within their own union. They applied, for each union, the Shapley value to the subgame within the union.

Owen (1977) proposed a different value for TU-games with coalition structure. He considered the situation in which all players in the game are allowed to cooperate, but a subset of players within a union can only cooperate with complete other unions. The Owen value for TU-games with coalition structure can be obtained by applying the Shapley value twice. First, to a game between the unions, assigning a value to each union, and then to a game within the union, distributing the value of a union among its players.

Recently, Kamijo (2011) has introduced a new value for TU-games with coalition structure. The main difference between this value and the Owen value is that unions are only allowed to cooperate when all players in the unions agree (i.e., only complete unions can cooperate). Kamijo's new approach provides a so-called restricted TU-game in which the worth of an arbitrary coalition of players is equal to the worth of the union of all complete unions within the coalition, plus the sum of the worths of all remaining parts of the coalition that are not complete unions. We call this restricted game the partition restricted game. The value proposed by Kamijo (2011) assigns to every game in coalition structure the Shapley value of the partition restricted game. Kamijo (2011) showed that his value is the unique value that satisfies the axioms of 'efficiency', 'balanced contributions' and

[1] Shapley (1953) describes a value as providing for each player an a priori assessment of the utility of becoming involved in a game.

'collective balanced contributions'.

A different form of restrictions on TU-games was considered by Myerson (1977). In his model the restrictions in the game are not given by a partition of the set of players but by the links in an undirected (communication) graph. Players are only allowed to cooperate in a coalition when they are connected in a graph, thus, when there exists a set of links (edges) in the graph that connects the cooperating players. TU-games with this kind of restrictions are known as TU-games with graph structure, or TU-games with graph restricted communication. The approach of Myerson (1977) gives a different kind of restricted game than the partition restricted game. The Myerson, or graph, restricted game is the TU-game in which the worth of a coalition is equal to the sum of the worths of its maximally connected subsets. The Myerson value is defined as the Shapley value of the graph restricted game. Myerson (1977) showed that his value is the unique value that satisfies the axioms of 'component efficiency' and 'fairness'.

Vázquez-Brage, García-Jurado and Carreras (1996) combined the ideas of Aumann and Drèze (1974) and Myerson (1977) in TU-games with coalition and graph structure, or TU-games with graph restricted communication and a priori unions. As a value for such games they proposed the Owen value (the value taking into account the partition into a priori unions) of the Myerson restricted game (the game taking into account the graph on the set of players). Alonso-Meijide, Álvarez-Mozos and Fiestras-Janeiro (2009) suggested two other values for TU-games with coalition and graph structure. They applied Banzhaf (1965) type modifications of the Owen value to the Myerson restricted game.

In this paper we propose two new values for TU-games with coalition and graph structure. They are obtained by applying the Shapley value to two restricted games associated with a TU-game with coalition and graph structure. The two restricted games combine the ideas of Myerson (1977) and Kamijo (2011). The first is called the *graph-partition restricted game* and is the partition restricted game of the graph restricted game. That is, first the graph structure is taken into account to obtain the graph restricted game, and then the partition structure is taken into account by taking the partition restricted game of the graph restricted game. The second is called the *partition-graph restricted game* and is obtained the other way around: it is the graph restricted game of the partition restricted game. It follows from Owen (1986) that for a partition-graph restricted game the (Harsanyi) dividend of any coalition that is not connected in the graph is zero.[2] For a graph-partition restricted game it is shown that the dividend of every coalition that is neither a subset of a union, nor a union of unions is zero. This implies that, in general,

[2]The (Harsanyi) dividend of a coalition is the additional contribution of cooperation among the players in a coalition, that they did not already realize by cooperating in smaller coalitions, see Harsanyi (1963).

the graph-partition restricted game is not equal to the partition-graph restricted game.

The two new values are defined as the Shapley values of the two types of restricted game. We show that the Shapley value of the graph-partition restricted game is characterized by the axioms of 'graph efficiency', 'balanced contributions' and 'collective balanced contributions'. The Shapley value of the partition-graph restricted game is characterized by the axioms of 'partition component efficiency' and 'fairness'.

To assess the usefulness of our two new values we apply them to an economic example. In this example the players can be viewed as cities that cooperate together within countries, being the a priori unions of the coalition structure, and the graph between the players can be viewed as a network of natural gas (or oil) pipelines between the cities.

The paper is organized as follows. In Section 2 we recall TU-games, TU-games with graph structure and TU-games with coalition structure. In Section 3 we introduce the two restricted games associated to a game with coalition and graph structure. In Section 4 we consider the two solutions that are obtained by applying the Shapley value to the two restricted games, and provide axiomatic characterizations. We also compare the characterizing sets of axioms with the axioms for the Myerson value (only taking into account the graph structure) and the value proposed in Kamijo (2011) (only taking into account the coalition structure). In Section 5 we apply the new values to an economic example and compare the outcomes to other values. Finally, we conclude in Section 6.

2 Cooperative games

2.1 TU-games

A cooperative game with transferable utility in characteristic function form, or TU-game, is a pair (N, v), where $N \subset \mathbb{N}$ is a finite set of $n = |N| \geq 2$ players (agents) and $v : 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. We denote the collection of all TU-games by $\mathcal{G}$. A subset $S \subseteq N$, $S \neq \emptyset$, is called a coalition. For any coalition S, $v(S)$ displays the worth of that coalition. The worth of a coalition can be interpreted as the wealth, measured in units of transferable utility, which the members of coalition S are able to divide among themselves when they decide to cooperate. For $S \subset N$, the game (S, v_S) denotes the subgame restricted to S with characteristic function $v_S(T) = v(T)$ for every $T \subseteq S$. For arbitrary $K \subset \mathbb{N}$, we denote $\mathbb{R}^K$ as the $|K|$-dimensional Euclidean space with elements $x \in \mathbb{R}^K$ having components x_i, $i \in K$.

A special class of TU-games is the class of unanimity games. For each nonempty $T \subseteq N$, the unanimity game (N, u^T) is given by the player set N and characteristic function $u^T(S) = 1$ if $T \subseteq S$, and $u^T(S) = 0$ otherwise. It is known that for every $(N, v) \in \mathcal{G}$, the characteristic function v can be written as a linear combination of the characteristic functions of the unanimity games (N, u^T) in a unique way: $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \Delta_v(T) u^T$, where $\Delta_v(T),\ T \in 2^N \setminus \{\emptyset\}$, are the (Harsanyi) dividends, see Harsanyi (1963). By definition of the unanimity games it follows that $v(S) = \sum_{T \subseteq S} \Delta_v(T)$, i.e., the worth $v(S)$ is equal to the dividend of S plus the sum of the dividends of all its proper subcoalitions. The dividend of S thus can be interpreted as the additional contribution of cooperation among the players in S, that they did not already realize by cooperating in smaller coalitions. Using the Möbius transform it follows that

$$\Delta_v(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S),\ T \in 2^N \backslash \{\emptyset\}.$$

A value f on $\mathcal{G}$ assigns a unique payoff vector $f(N, v) \in \mathbb{R}^N$ to every TU-game $(N, v) \in \mathcal{G}$. A value is efficient if it distributes $v(N)$, thus, if $\sum_{i \in N} f_i(N, v) = v(N)$ for every $(N, v) \in \mathcal{G}$. The best-known efficient value is the Shapley value (Shapley, 1953). This value, denoted by Sh, equally distributes the dividends among the players in the corresponding coalitions: for every TU-game $(N, v) \in \mathcal{G}$,

$$Sh_i(N, v) = \sum_{\{T \subseteq N \ |\ i \in T\}} \frac{\Delta_v(T)}{|T|}\ ,\ i \in N.$$

2.2 TU-games with graph structure

A graph is a pair (N, L) where N is a set of nodes and $L \subseteq \{\{i, j\} | i, j \in N, i \neq j\}$ is a set of unordered pairs of distinct elements of N. In this paper the nodes represent the players in a game (N, v). We therefore refer to them as players. The elements of L are called links or edges. If there is no confusion about the player set N, we will write a graph (N, L) just by its set of links L. We denote the set of all graphs on N by $\mathcal{L}^N$. For $S \subseteq N$, the graph $(S, L(S))$ with $L(S) = \{\{i, j\} \in L | i, j \in S\}$ is called the subgraph of L on S. Given $L \in \mathcal{L}^N$, a sequence of k different players $(i_1, ..., i_k)$ is a path in $L(S)$ if $\{i_l, i_{l+1}\} \in L(S)$ for $l = 1, ..., k-1$. Two players $i, j \in S$ are called connected in $L(S)$ if $i = j$ or there exists a path $(i_1, ..., i_k)$ in $L(S)$ with $i_1 = i$ and $i_k = j$. A coalition $S \subseteq N$ is said to be a connected coalition (or connected in L) if every two players in S are connected in $L(S)$. A coalition $K \subseteq N$ is a component of (N, L) if and only if (i) K is connected in L, and (ii) $K \cup \{i\}$ is not connected in L for every $i \in N \backslash K$. The set of components of $(S, L(S))$ is denoted by $C^L(S)$. Note that every player in $S \subseteq N$ that is not linked with any other player in S is a (singleton) component in $(S, L(S))$.

A TU-game with graph structure is a triple (N, v, L) with $(N, v) \in \mathcal{G}$ and $L \in \mathcal{L}^N$ a graph on N. We denote the collection of all TU-games with graph structure (N, v, L) by $\mathcal{G}_G$. Following Myerson (1977), in a game with graph structure $(N, v, L) \in \mathcal{G}_G$, a coalition S is only able to realize its worth $v(S)$ if S is connected in L. When S is not connected in L, the players in S can realize the sum of the worths of the components of the subgraph $(S, L(S))$. Given a TU-game $(N, v) \in \mathcal{G}$ and a graph $L \in \mathcal{L}^N$, the Myerson or graph restricted game induced by L is the TU-game $(N, v^L) \in \mathcal{G}$ with player set N and characteristic function

$$v^L(S) = \sum_{T \in C^L(S)} v(T) \text{ for all } S \subseteq N.$$

A value f on $\mathcal{G}_G$ assigns a unique payoff vector $f(N, v, L) \in \mathbb{R}^N$ to every $(N, v, L) \in \mathcal{G}_G$. The Myerson value (Myerson, 1977) of a TU-game with graph structure, denoted by My, is defined as the value that assigns to every $(N, v, L) \in \mathcal{G}_G$ the Shapley value of the corresponding graph restricted game (N, v^L). That is, for every $(N, v, L) \in \mathcal{G}_G$ the Myerson value is defined as $My(N, v, L) = Sh(N, v^L)$. Myerson (1977) axiomatized this value for games with graph structure by 'component efficiency' and 'fairness'.

2.3 TU-games with coalition structure

Let $\mathcal{P}^N$ be the set of partitions of N. So, for some $m \leq |N|$, $P = \{P_1, ..., P_m\} \in \mathcal{P}^N$ if and only if (i) $\bigcup_{i=1}^m P_i = N$, (ii) $P_k \neq \emptyset$ for all $k \in \{1, \ldots, m\}$, and (iii) $P_k \cap P_l = \emptyset$ for all $k, l \in \{1, \ldots, m\}$ with $k \neq l$. For a given $P = \{P_1, ..., P_m\} \in \mathcal{P}^N$, let $M = \{1, ..., m\}$. Then $P = \{P_j | j \in M\}$ is called a coalition structure, or a system of a priori unions, and any element P_j, $j \in M$, is called a union of P. A TU-game with coalition structure is a triple (N, v, P) with $(N, v) \in \mathcal{G}$ and $P \in \mathcal{P}^N$ a partition of N. We denote the collection of all TU-games with coalition structure by $\mathcal{G}_C$. A value f on $\mathcal{G}_C$ assigns a unique payoff vector $f(N, v, P) \in \mathbb{R}^N$ to every TU-game with coalition structure $(N, v, P) \in \mathcal{G}_C$.

Aumann and Drèze (1974) assume that every union in P acts as a stand-alone coalition. One can obtain the Aumann-Drèze value by applying to every P_j, $j \in M$, the Shapley value to the subgame (P_j, v_{P_j}). By efficiency of the Shapley value, the total payoff assigned to the players in P_j is equal to $v(P_j)$. Since in general $\sum_{j \in M} v(P_j) \neq v(N)$, the Aumann-Dréze value is not efficient.

The best-known efficient value for games with coalition structure is the Owen value, Owen (1977). One can obtain the Owen value of a TU-game with coalition structure by applying the Shapley value twice.[3] To do so, first define the quotient game of (N, v, P) as the game (M, v^P) with player set M and characteristic function $v^P(Q) = v(\cup_{h \in Q} P_h)$ for every $Q \subseteq M$. Then, consider the TU-game (P_k, v^k) with player set P_k and characteristic function v^k which is obtained by assigning to every coalition $S \subseteq P_k$ the Shapley value

[3]See also van den Brink and van der Laan (2005) in which Owen-type values for the class of games with coalition structures are given that determine the individual payoff shares as the multiplicative product of two shares in the total payoff.

payoff of player $k \in M$ in a modified quotient game in which P_k is replaced by $S \subseteq P_k$. Next, for each $k \in M$, the Shapley value is applied to the game (P_k, v^k) to obtain the Owen payoffs of the players $i \in P_k$: $Ow_i(N, v, P) = Sh_i(P_k, v^k)$, $i \in P_k$, $k \in M$. The Owen value is efficient, because by efficiency of the Shapley value we have that $\sum_{i \in P_k} Ow_i(N, v, P) = v^k(P_k) = Sh_k(M, v^P)$ and $\sum_{k \in M} Sh_k(M, v^P) = v(N)$.

In this paper we follow the approach of Kamijo (2011) by assuming that individual players are able to cooperate within their union, but need their full union in order to cooperate with players from outside their union. Thus, given a partition $P \in \mathcal{P}^N$, players in any coalition $S \subseteq P_j \in P$ can cooperate with each other and obtain the worth of the coalition $v(S)$. In addition, there is the possibility of cooperation among players in different unions, but only if all players in these unions cooperate. Let $S \subset P_j \in P$ and $P_k \in P$, $P_k \neq P_j$. While P_j and P_k can obtain their worth $v(P_j \cup P_k)$ when they decide to cooperate, S and P_k can obtain only $v(S) + v(P_k)$ because all players in P_j and P_k are necessary in establishing cooperation between these unions.

Given $P = \{P_j \mid j \in M\} \in \mathcal{P}^N$, for all $S \subseteq N$, $S \neq \emptyset$, denote

$$S/P = \{\cup_{\{k \in M | S \cap P_k = P_k\}} P_k\} \cup \{S \cap P_k \mid S \cap P_k \neq P_k,\ k \in M\}.$$

Hence, S/P is a collection of disjunct sets with as elements the union of all complete unions P_k that are contained in S, and the sets $S \cap P_k$, for every union that contains players outside S. Given a TU-game $(N, v) \in \mathcal{G}$ and a partition $P \in \mathcal{P}^N$, the partition restricted game induced by coalition structure P is the TU-game $(N, v|_P)$ with player set N and characteristic function

$$v|_P(S) = \sum_{T \in S/P} v(T), \quad \text{for all} \quad S \subseteq N.$$

The value for TU-games with coalition structure proposed by Kamijo (2011), called the collective value and denoted by Ka, assigns to every TU-game with coalition structure (N, v, P) the Shapley value of the corresponding partition restricted game $(N, v|_P)$. Thus, for every $(N, v, P) \in \mathcal{G}_C$ the collective value is defined by $Ka(N, v, P) = Sh(N, v|_P)$. Kamijo (2011) axiomatized this value by 'efficiency', 'balanced contributions' and 'collective balanced contributions'.

3 The graph-partition and partition-graph restricted games

A *TU-game with coalition and graph structure* is a quadruple (N, v, L, P) with $(N, v) \in \mathcal{G}$ a TU-game, $L \in \mathcal{L}^N$ a graph and $P \in \mathcal{P}^N$ a partition of N. We denote the collection of all TU-games with coalition and graph structure by $\mathcal{G}_{CG}$.

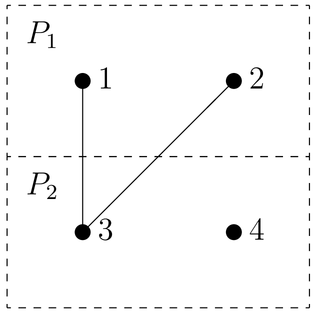

Figure 1: $N = \{1, 2, 3, 4\}$, $L = \{\{1, 3\}, \{2, 3\}\}$, $P = \{\{1, 2\}, \{3, 4\}\}$

With each TU-game with coalition and graph structure we associate two restricted TU-games. These restricted games take into account both the cooperation restrictions arising from the partition as well as those arising from the graph. First, we define the *graph-partition restricted game* induced by L and P. This game associates to every $(N, v, L, P) \in \mathcal{G}_{CG}$ the corresponding TU-game $(N, v^L|_P)$. So, given $(N, v, L, P) \in \mathcal{G}_{CG}$, the graph-partition restricted game is obtained by first taking the graph restricted game v^L of (N, v, L) and then the partition restricted game of (N, v^L, P). Second, the *partition-graph restricted game* is defined the other way around and associates to every $(N, v, L, P) \in \mathcal{G}_{CG}$ the corresponding TU-game $(N, (v|_P)^L)$. So, given $(N, v, L, P) \in \mathcal{G}_{CG}$, the partition-graph restricted game is obtained by first taking the partition restricted game $v|_P$ of (N, v, P) and then the graph restricted game of $(N, v|_P, L)$.

In general, the game $(N, v^L|_P)$ can differ from the game $(N, (v|_P)^L)$, and thus the order in which we apply the cooperation restrictions matters. This is illustrated in the next example.

Example 3.1 Let $(N, v, L, P) \in \mathcal{G}_{CG}$ be such that $N = \{1, 2, 3, 4\}$, $L = \{\{1, 3\}, \{2, 3\}\}$ and $P = \{\{1, 2\}, \{3, 4\}\}$, as displayed in Figure 1. Then $v^L|_P(S)$ and $(v|_P)^L(S)$, $S \subseteq N$, are as given in Tables 3.2 and 3.3. In the last column of both tables the dividends are given. For readability, in the tables we write $v(\{i, \ldots, j\})$ as $v(i, \ldots, j)$. $\square$

S	$v^L(S)$	$v^L\|_P(S)$			$\Delta_{v^L\|_P}(S)$
$\emptyset$	$v(\emptyset)$	$v^L(\emptyset)$	$=$	$v(\emptyset)$	0
1	$v(1)$	$v^L(1)$	$=$	$v(1)$	$v(1)$
2	$v(2)$	$v^L(2)$	$=$	$v(2)$	$v(2)$
3	$v(3)$	$v^L(3)$	$=$	$v(3)$	$v(3)$
4	$v(4)$	$v^L(4)$	$=$	$v(4)$	$v(4)$
$1,2$	$v(1)+v(2)$	$v^L(1,2)$	$=$	$v(1)+v(2)$	0
$1,3$	$v(1,3)$	$v^L(1)+v^L(3)$	$=$	$v(1)+v(3)$	0
$1,4$	$v(1)+v(4)$	$v^L(1)+v^L(4)$	$=$	$v(1)+v(4)$	0
$2,3$	$v(2,3)$	$v^L(2)+v^L(3)$	$=$	$v(2)+v(3)$	0
$2,4$	$v(2)+v(4)$	$v^L(2)+v^L(4)$	$=$	$v(2)+v(4)$	0
$3,4$	$v(3)+v(4)$	$v^L(3,4)$	$=$	$v(3)+v(4)$	0
$1,2,3$	$v(1,2,3)$	$v^L(1,2)+v^L(3)$	$=$	$v(1)+v(2)+v(3)$	0
$1,2,4$	$v(1)+v(2)+v(4)$	$v^L(1,2)+v^L(4)$	$=$	$v(1)+v(2)+v(4)$	0
$1,3,4$	$v(1,3)+v(4)$	$v^L(1)+v^L(3,4)$	$=$	$v(1)+v(3)+v(4)$	0
$2,3,4$	$v(2,3)+v(4)$	$v^L(2)+v^L(3,4)$	$=$	$v(2)+v(3)+v(4)$	0
N	$v(1,2,3)+v(4)$	$v^L(N)$	$=$	$v(1,2,3)+v(4)$	$v(1,2,3)-v(1)$ $-v(2)-v(3)$

Table 3.2 Characteristic function and dividends of $v^L|_P$.

S	$v\|_P(S)$	$(v\|_P)^L(S)$			$\Delta_{(v\|_P)^L}(S)$
$\emptyset$	$v(\emptyset)$	$v\|_P(\emptyset)$	$=$	$v(\emptyset)$	0
1	$v(1)$	$v\|_P(1)$	$=$	$v(1)$	$v(1)$
2	$v(2)$	$v\|_P(2)$	$=$	$v(2)$	$v(2)$
3	$v(3)$	$v\|_P(3)$	$=$	$v(3)$	$v(3)$
4	$v(4)$	$v\|_P(4)$	$=$	$v(4)$	$v(4)$
$1,2$	$v(1,2)$	$v\|_P(1)+v\|_P(2)$	$=$	$v(1)+v(2)$	0
$1,3$	$v(1)+v(3)$	$v\|_P(1,3)$	$=$	$v(1)+v(3)$	0
$1,4$	$v(1)+v(4)$	$v\|_P(1)+v\|_P(4)$	$=$	$v(1)+v(4)$	0
$2,3$	$v(2)+v(3)$	$v\|_P(2,3)$	$=$	$v(2)+v(3)$	0
$2,4$	$v(2)+v(4)$	$v\|_P(2)+v\|_P(4)$	$=$	$v(2)+v(4)$	0
$3,4$	$v(3,4)$	$v\|_P(3)+v\|_P(4)$	$=$	$v(3)+v(4)$	0
$1,2,3$	$v(1,2)+v(3)$	$v\|_P(1,2,3)$	$=$	$v(1,2)+v(3)$	$v(1,2)$ $-v(1)-v(2)$
$1,2,4$	$v(1,2)+v(4)$	$v\|_P(1)+v\|_P(2)+v\|_P(4)$	$=$	$v(1)+v(2)+v(4)$	0
$1,3,4$	$v(1)+v(3,4)$	$v\|_P(1,3)+v\|_P(4)$	$=$	$v(1)+v(3)+v(4)$	0
$2,3,4$	$v(2)+v(3,4)$	$v\|_P(2,3)+v\|_P(4)$	$=$	$v(2)+v(3)+v(4)$	0
N	$v(N)$	$v\|_P(1,2,3)+v\|_P(4)$	$=$	$v(1,2)+v(3)+v(4)$	0

Table 3.3 Characteristic function and dividends of $(v|_P)^L$.

Given a game with graph structure $(N, v, L) \in \mathcal{G}_G$, Owen (1986) has shown that for the corresponding graph restricted game (N, v^L) the dividend $\Delta_{v^L}(S)$ is equal to zero for any coalition S that is not connected in L. Since the partition-graph restricted game is defined as the graph restricted game of the partition restricted game, we have that in a partition-graph restricted game the dividend of any coalition that is not connected in the graph is zero.

Corollary 3.4 *For every $(N, v, L, P) \in \mathcal{G}_{CG}$ and $S \in 2^N \backslash \{\emptyset\}$, if S is not connected in L then $\Delta_{(v|_P)^L}(S) = 0$.*

The corollary does not hold for the graph-partition restricted game $(N, v^L|_P)$. For instance, in Example 3.1 we have that $S = N$ is not connected, but $\Delta_{v^L|_P}(N) = v(\{1,2,3\}) - v(\{1\}) - v(\{2\}) - v(\{3\})$. To find the counterpart of Corollary 3.4 for $(N, v^L|_P)$, we first consider games with coalition structure $(N, v, P) \in \mathcal{G}_C$. For a fixed player set N, let $\mathcal{G}^N$ denote the collection of all TU-games on N. Then, for $P = \{P_j \mid j \in M\} \in \mathcal{P}^N$, define the mapping $Z_P \colon \mathcal{G}^N \to \mathcal{G}^N$ by

$$Z_P(v) = v|_P.$$

So, Z_P maps each characteristic function $v \in \mathcal{G}^N$ to characteristic function $v|_P \in \mathcal{G}^N$. Because the elements of the collection S/P are fixed, Z_P is a linear mapping. In order to investigate the behavior of the mapping Z_P we consider the images of the unanimity games (N, u^T). It is not hard to see that if there is a $j \in M$ with $T \subseteq P_j$, or there is a $Q \subseteq M$ such that $T = \cup_{q \in Q} P_q$, then $Z_P(u^T) = u^T$. But, if T is not of this form, then we have $u^T|_P(S) = \sum_{R \in S/P} u^T(R) = \sum_{\{R \in S/P | T \subseteq R\}} u^T(R)$ for all $S \in 2^N \backslash \{\emptyset\}$. Hence, $Z_P(u^T) = d^T$, where d^T is the game given by

$$d^T(S) = \begin{cases} 1 & \text{if there is an } R \in S/P \text{ such that } T \subseteq R \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Since $v = \sum_{S \in 2^N \backslash \{\emptyset\}} \Delta_v(S)\, u^S$ for any game $(N, v) \in \mathcal{G}$, it holds that

$$d^T = \sum_{S \in 2^N \backslash \{\emptyset\}} \Delta_{d^T}(S) u^S.$$

Note that $\Delta_{d^T}(S) = 0$ unless $T \subseteq S$. In addition we have the next proposition.

Proposition 3.5 *Let $P = \{P_j | j \in M\}$ be a partition of N and $S \subseteq N$. If there is no $j \in M$ with $S \subseteq P_j$, and there is no $Q \subseteq M$ such that $S = \bigcup_{q \in Q} P_q$, then $\Delta_{d^T}(S) = 0$ for all $T \in 2^N \backslash \{\emptyset\}$.*

Proof. The proof proceeds along the same lines as the proof of Theorem 2 in Owen (1986). Let $S \in 2^N \backslash \{\emptyset\}$ be such that there is no $j \in M$ with $S \subseteq P_j$, and there is no $Q \subseteq M$ with $S = \bigcup_{q \in Q} P_q$. Let $T \in 2^N \backslash \{\emptyset\}$ be arbitrary.

If $T \not\subseteq S$, then $d^T(H) = 0$ for all $H \subseteq S$. So, $\Delta_{d^T}(S) = 0$. Also, if $T \subseteq S$ but there is no $R \in S/P$ with $T \subseteq R \subseteq S$, then $d^T(H) = 0$ for all $H \subseteq S$. Again, $\Delta_{d^T}(S) = 0$.

Next, suppose that $T \subseteq S$ and there is $R \in S/P$ with $T \subseteq R \subseteq S$. For $H \subseteq S$, write $H = H_1 \cup H_2$ with $H_1 \subseteq R$ and $H_2 \subseteq S \backslash R$. It is not difficult to see that $d^T(H) = d^T(H_1)$. Then,

$$\begin{aligned} \Delta_{d^T}(S) &= \sum_{H \subseteq S} (-1)^{|S|-|H|} d^T(H) \\ &= \sum_{H_1 \subseteq R} \sum_{H_2 \subseteq S \backslash R} (-1)^{|R|-|H_1|} (-1)^{|S|-|R|-|H_2|} d^T(H_1) \\ &= \sum_{H_1 \subseteq R} (-1)^{|R|-|H_1|} d^T(H_1) \Big[\sum_{H_2 \subseteq S \backslash R} (-1)^{|S|-|R|-|H_2|} \Big] \\ &= \sum_{H_1 \subseteq R} (-1)^{|R|-|H_1|} d^T(H_1) \Big[\sum_{r_2=0}^{|S|-|R|} (-1)^{|S|-|R|-r_2} \binom{|S|-|R|}{r_2} \Big], \end{aligned} \tag{3.1}$$

where the last equality follows because $S \backslash R$ has $\binom{|S|-|R|}{r_2}$ subsets of cardinality r_2. A lemma in Owen (1986), that follows directly from the binomial expansion of $(-1+1)^n$, states that for any integer $n \geq 0$,

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} = \begin{cases} 0 & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases} \tag{3.2}$$

Because there is no $j \in M$ with $S \subseteq P_j$, and there is no $Q \subseteq M$ with $S = \bigcup_{q \in Q} P_q$, and $R \in S/P$ (so that there is a $j \in M$ with $R \subseteq P_j$, or there is a $Q \subseteq M$ with $R = \bigcup_{q \in Q} P_q$) it holds that $S \backslash R \neq \emptyset$, and thus $|S| - |R| \geq 1$. It then follows from (3.2) that the last bracket in equation (3.1) is zero. It can be concluded that $\Delta_{d^T}(S) = 0$. □

Let $\mathcal{S}$ be defined by

$$\mathcal{S} = \{S \subseteq N \mid S \subseteq P_j \text{ for some } j \in M\} \,\cup\, \{S \subseteq N \mid S = \cup_{q \in Q} P_q \text{ for some } Q \subseteq M\}.$$

Then Proposition 3.5 leads to the next theorem.

Theorem 3.6 *Let $P = \{P_j | j \in M\}$ be a partition of N. Then the unanimity games u^S, $S \in \mathcal{S}$, form a basis for the image of Z_P.*

Proof. It follows from Proposition 3.5 that the image $Z_P(u^T)$ of any unanimity game u^T is a linear combination of unanimity games u^S, $S \in \mathcal{S}$. Additionally, if $S \in \mathcal{S}$, then u^S is its own image. This implies that the u^S, $S \in \mathcal{S}$, span the image space and, because they are independent, form a basis for it. $\square$

It follows from Theorem 3.6 that $\Delta_{v|_P}(S) = 0$ for any coalition S such that there is no $j \in M$ with $S \subseteq P_j$, and there is no $Q \subseteq M$ such that $S = \bigcup_{q \in Q} P_q$. That is, any coalition that is neither a subset of a union, nor a union of unions has a zero dividend in the partition-restricted TU-game. Since the graph-partition restricted game is defined as the partition restricted game of the graph restricted game, we have that in a graph-partition restricted game the dividend of any coalition that is neither a subset of a union, nor a union of unions is zero.

Corollary 3.7 *For every $(N, v, L, P) \in \mathcal{G}_{CG}$ and $S \in 2^N \backslash \{\emptyset\}$, if there is no $j \in M$ with $S \subseteq P_j$, and there is no $Q \subseteq M$ such that $S = \bigcup_{q \in Q} P_q$, then $\Delta_{v^L|_P}(S) = 0$.*

This corollary does not hold for the partition-graph restricted game $(N, (v|_P)^L)$. For instance, in Example 3.1 we have that $S = \{1, 2, 3\}$ is not a subset of any P_j and $S \neq \bigcup_{q \in Q} P_q$ for all $Q \subseteq M$. However, $\Delta_{(v|_P)^L}(\{1, 2, 3\}) = v(\{1, 2\}) - v(\{1\}) - v(\{2\})$.

4 Two values for TU-games with coalition and graph structure

A value f on $\mathcal{G}_{CG}$ assigns a unique payoff vector $f(N, v, L, P) \in \mathbb{R}^n$ to every TU-game with coalition and graph structure $(N, v, L, P) \in \mathcal{G}_{CG}$. We introduce two new values for TU-games with coalition and graph structure by applying the Shapley value to the two restricted games defined in the previous section.

Definition 4.1
1. *The* graph-partition value *on the class of TU-games with coalition and graph structure is the value ϕ assigning to every $(N, v, L, P) \in \mathcal{G}_{CG}$ the payoff vector $\phi(N, v, L, P) = Sh(N, v^L|_P)$.*
2. *The* partition-graph value *on the class of TU-games with coalition and graph structure is the value ψ assigning to every $(N, v, L, P) \in \mathcal{G}_{CG}$ the payoff vector $\psi(N, v, L, P) = Sh(N, (v|_P)^L)$.*

Note that $\phi(N, v, L, P) = Ka(N, v^L, P)$ and that $\psi(N, v, L, P) = My(N, v|_P, L)$. Because $v^L|_P$ does not have to be equal to $(v|_P)^L$, in general $\phi(N, v, L, P)$ is not equal to $\psi(N, v, L, P)$.

Example 4.2 Let $(N, v, L, P) \in \mathcal{G}_{CG}$ be as in Example 3.1. From the dividends derived in that example it follows straightforwardly that the graph-partition value is given by

$$\phi_i(N, v, L, P) = v(\{i\}) + \frac{1}{4}[v(\{1, 2, 3\}) - v(\{1\}) - v(\{2\}) - v(\{3\})],\ i = 1, 2, 3, 4$$

and that the partition-graph value is given by

$$\psi_j(N, v, L, P) = v(\{j\}) + \frac{1}{3}[v(\{1, 2\}) - v(\{1\}) - v(\{2\})],\ j = 1, 2, 3,$$

and $\psi_4(N, v, L, P) = v(\{4\})$. □

It is obvious that for special structures we obtain the Myerson value or Kamijo's collective value. By definition, the Myerson value only takes into account the graph structure and ignores the coalition structure. Therefore the graph-partition value is equal to the Myerson value when $P = \{N\}$.

Proposition 4.3 *Let $(N, v, L, P) \in \mathcal{G}_{CG}$. If $P = \{N\}$ then $\phi(N, v, L, P) = My(N, v, L)$.*

The collective value of Kamijo (2011) only takes into account the coalition structure and ignores the graph structure. Therefore the partition-graph value is equal to the collective value when L is the complete graph.

Proposition 4.4 *Let $(N, v, L, P) \in \mathcal{G}_{CG}$. If $L = \{\{i, j\} \mid i, j \in N,\ i \neq j\}$ then $\psi(N, v, L, P) = Ka(N, v, P)$.*

The two values defined above for games with coalition and graph structure are based on two different restricted games. Both restricted games are obtained by applying the methods of Myerson (1977) and Kamijo (2011), but in different orders. To characterize these two values we generalize the axiomatizations of the Myerson value for graph restricted games (Myerson, 1977) and the collective value for partition restricted games (Kamijo, 2011) to the class of TU-games with coalition and graph structure.

4.1 Characterization of the graph-partition value

The efficiency property as implicitly used in Kamijo (2011) states that players in N distribute the worth $v(N)$ among themselves. Here we formulate this axiom in the context of games with coalition and graph structure.

Axiom 4.5 Efficiency
A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ is efficient if for any $(N, v, L, P) \in \mathcal{G}_{CG}$ it holds that $\sum_{i \in N} f_i(N, v, L, P) = v(N)$.

The graph-partition value does not satisfy this axiom in general. However, it satisfies a weaker version stating that the players in N distribute the sum of the worths of the connected components of L among themselves. This takes into account that in a game with graph structure players can only cooperate when they are connected in the graph.

Axiom 4.6 Graph efficiency
A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ is graph efficient if for any $(N, v, L, P) \in \mathcal{G}_{CG}$ it holds that $\sum_{i \in N} f_i(N, v, L, P) = \sum_{K \in C^L(N)} v(K)$.

Clearly, when N is connected in L, then for every solution f satisfying graph efficiency it holds that $\sum_{i \in N} f_i(N, v, L, P) = v(N)$.

Next we generalize the balanced contributions axiom for TU-games with coalition structure used in Kamijo (2011) to the setting of TU-games with coalition and graph structure. It states that, given that the coalition structure is given by $P = \{N\}$, the loss in value that player $i \in N$ experiences when player $j \in N$ leaves the game is equal to the loss that player j experiences when player i leaves the game.[4] For convenience, for every $j \in N$ we denote $N_{-j} = N \setminus \{j\}$, $v_{-j} = v_{N \setminus \{j\}}$ and $L_{-j} = L(N \setminus \{j\})$.

Axiom 4.7 Balanced contributions
A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ satisfies balanced contributions if for any $(N, v, L, \{N\}) \in \mathcal{G}_{CG}$ it holds that

$$f_i(N, v, L, \{N\}) - f_i(N_{-j}, v_{-j}, L_{-j}, \{N_{-j}\}) = f_j(N, v, L, \{N\}) - f_j(N_{-i}, v_{-i}, L_{-i}, \{N_{-i}\}),$$

for all $i, j \in N$.

The following collective balanced contributions axiom is a generalization to the setting of TU-games with coalition and graph structure of the collective balanced contributions axiom for TU-games with coalition structure of Kamijo (2011). It states that, given two different unions P_k and P_h in P, for every $i \in P_k$ and $j \in P_h$, the loss in value that player i experiences when union $P_h \in P$ leaves the game is equal to the loss in value that player j experiences when union $P_k \in P$ leaves the game.[5] Again for convenience, for every $h \in M$ we denote $N_{-P_h} = N \setminus P_h$, $v_{-P_h} = v_{N \setminus P_h}$, $L_{-P_h} = L(N \setminus P_h)$ and $P_{-P_h} = P \setminus \{P_h\}$.

[4] Note that Myerson (1980) defined balanced contributions for conference structures on a fixed player set. In his model, instead of a player leaving the game, all feasible coalitions containing this player are no longer feasible but, by definition, the player stays connected as a singleton.

[5] Although it is easy to formulate balanced contributions for a fixed player set, it is more difficult to state collective balanced contributions on a fixed player set since we need to specify how the players in P_h 'stay' in the game. Therefore, these axioms are more different than their name might suggest.

Axiom 4.8 Collective balanced contributions
A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ satisfies collective balanced contributions if for any $(N, v, L, P) \in \mathcal{G}_{CG}$ with $|P| \geq 2$ it holds that

$$f_i(N, v, L, P) - f_i(N_{-P_h}, v_{-P_h}, L_{-P_h}, P_{-P_h}) = f_j(N, v, L, P) - f_j(N_{-P_k}, v_{-P_k}, L_{-P_k}, P_{-P_k})$$

for every two different unions P_k and P_h in P, and all $i \in P_k \in P$, and all $j \in P_h \in P$.

The axioms 4.6-4.8 characterize the graph-partition value.

Theorem 4.9 *A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ satisfies graph efficiency, balanced contributions and collective balanced contributions if and only if $f(N, v, L, P) = \phi(N, v, L, P)$ for every $(N, v, L, P) \in \mathcal{G}_{CG}$.*

Proof. First, we show that ϕ satisfies graph efficiency, balanced contributions and collective balanced contributions. Graph efficiency follows from

$$\sum_{i \in N} \phi_i(N, v, L, P) = \sum_{i \in N} Sh_i(N, v^L|_P) = \sum_{i \in N} \sum_{\{S \subseteq N | i \in S\}} \frac{\Delta_{v^L|_P}(S)}{|S|} =$$

$$\sum_{S \subseteq N} \Delta_{v^L|_P}(S) = v^L|_P(N) = \sum_{S \in N/P} v^L(S) = v^L(N) = \sum_{K \in C^L(N)} v(K),$$

where the first, second, fifth and seventh equalities follow by definition, the third by rearranging terms, the fourth by the expression for the dividends and the sixth because $N/P = N$.

Next, for every pair $i, j \in N$ it holds that

$$\begin{aligned}
&\phi_i(N, v, L, \{N\}) - \phi_i(N_{-j}, v_{-j}, L_{-j}, \{N_{-j}\}) = \\
&\quad Sh_i(N, v^L|_{\{N\}}) - Sh_i(N_{-j}, (v_{-j})^{L_{-j}}|_{\{N_{-j}\}}) = Sh_i(N, v^L) - Sh_i(N_{-j}, (v_{-j})^{L_{-j}}) = \\
&\quad My_i(N, v, L) - My_i(N_{-j}, v_{-j}, L_{-j}) = My_j(N, v, L) - My_j(N_{-i}, v_{-i}, L_{-i}) = \\
&\quad Sh_j(N, v^L) - Sh_j(N_{-i}, (v_{-i})^{L_{-i}}) = Sh_j(N, v^L|_{\{N\}}) - Sh_j(N_{-i}, (v_{-i})^{L_{-i}}|_{\{N_{-i}\}}) = \\
&\qquad \phi_j(N, v, L, \{N\}) - \phi_j(N_{-i}, v_{-i}, L_{-i}, \{N_{-i}\}),
\end{aligned}$$

where the fourth equality follows because the value of Myerson (1977) satisfies balanced contributions for TU-games with graph structure[6] and all the others follow by definition. Hence, ϕ satisfies balanced contributions.

[6]This follows similar as shown in Myerson (1980) for a fixed player set.

Finally, given $L \in \mathcal{L}^N$ and $P_j \in P \in \mathcal{P}^N$ consider the TU-games $(N_{-P_j}, (v_{-P_j})^{L_{-P_j}})$ and $(N_{-P_j}, (v^L)_{-P_j})$. Because for all $S \subseteq N \setminus P_j$ it holds that

$$(v_{-P_j})^{L_{-P_j}}(S) = \sum_{T \in C^{L_{-P_j}}(S)} v_{-P_j}(T) = \sum_{T \in C^{L_{-P_j}}(S)} v(T) = v^{L_{-P_j}}(S) = v^L(S) = (v^L)_{-P_j}(S),$$

these games are equal. Now consider any $(N, v, L, P) \in \mathcal{G}_{CG}$ with $|P| \geq 2$ and take any $i \in P_k \in P$ and any $j \in P_l \in P$, $P_k \neq P_l$. Then,

$$\phi_i(N, v, L, P) - \phi_i(N_{-P_l}, v_{-P_l}, L_{-P_l}, P_{-P_l}) =$$

$$Ka_i(N, v^L, P) - Ka_i(N_{-P_l}, (v_{-P_l})^{L_{-P_l}}, P_{-P_l}) = Ka_i(N, v^L, P) - Ka_i(N_{-P_l}, (v^L)_{-P_l}, P_{-P_l}) =$$

$$Ka_j(N, v^L, P) - Ka_j(N_{-P_k}, (v^L)_{-P_k}, P_{-P_k}) = Ka_j(N, v^L, P) - Ka_j(N_{-P_k}, (v_{-P_k})^{L_{-P_k}}, P_{-P_k}) =$$

$$\phi_j(N, v, L, P) - \phi_j(N_{-P_k}, v_{-P_k}, L_{-P_k}, P_{-P_k}),$$

where the first and last equality follow by definition, the second and fourth because the TU-games in the expressions are equal and the third because the value of Kamijo (2011) satisfies collective balanced contributions for TU-games with coalition structure (see Kamijo (2011)). Hence, ϕ satisfies collective balanced contributions.

Second, we show that there can be at most one value that satisfies graph efficiency, balanced contributions and collective balanced contributions. Suppose that f satisfies these axioms and consider first all games $(N, v, L, P) \in \mathcal{G}_{CG}$ with $P = \{N\}$. We uniquely determine $f(N, v, L, P)$ for these games by induction on the number of players n. When $n = 1$ it follows directly from graph efficiency that $f_i(\{i\}, v, L, \{\{i\}\}) = v(\{i\}) = \phi_i(\{i\}, v, L, \{\{i\}\})$, $i \in N$. Next, suppose that f has been uniquely determined for all games $(K, v, L, P) \in \mathcal{G}_{CG}$ with $P = \{K\}$ and $|K| \leq n - 1$. Then applying the balanced contributions property to f for $(N, v, L, P) \in \mathcal{G}_{CG}$ with $P = \{N\}$ and $|N| = n$ gives

$$f_i(N, v, L, \{N\}) - f_i(N_{-j}, v_{-j}, L_{-j}, \{N_{-j}\}) = f_j(N, v, L, \{N\}) - f_j(N_{-i}, v_{-i}, L_{-i}, \{N_{-i}\})$$

for all $i, j \in N$. Notice that for all $i, j \in N$, the values $f_i(N_{-j}, v_{-j}, L_{-j}, \{N_{-j}\})$ and $f_j(N_{-i}, v_{-i}, L_{-i}, \{N_{-i}\})$ are known by the induction hypothesis. For some particular $i \in N$, say $i = i^0$, there are $n - 1$ equations of this type with $i = i^0$. Together with the efficiency equation

$$\sum_{i \in N} f_i(N, v, L, \{N\}) = v(N)$$

these form a system of $(n-1)+1 = n$ linearly independent equations and so these equations uniquely determine $f_i(N, v, L, \{N\})$, $i \in N$.

Finally, consider all games $(N, v, L, P) \in \mathcal{G}_{CG}$ with $|P| \geq 2$. Suppose that there are two different values f^1 and f^2 that both satisfy graph efficiency, balanced contributions and collective balanced contributions. Let P be a partition with a minimum number of unions (elements of P) such that $f^1(N, v, L, P) \neq f^2(N, v, L, P)$. It follows by the minimality of P that if P_h is any element of P, then $f^1(N_{-P_h}, v_{-P_h}, L_{-P_h}, P_{-P_h}) = f^2(N_{-P_h}, v_{-P_h}, L_{-P_h}, P_{-P_h})$. Now, by collective balanced contributions (and rearranging terms) we have for all $i \in P_k \in P$ and all $j \in P_l \in P$, $P_k \neq P_l$, that

$$f_i^1(N, v, L, P) - f_j^1(N, v, L, P) = f_i^1(N_{-P_l}, v_{-P_l}, L_{-P_l}, P_{-P_l}) - f_j^1(N_{-P_k}, v_{-P_k}, L_{-P_k}, P_{-P_k}) =$$

$$f_i^2(N_{-P_l}, v_{-P_l}, L_{-P_l}, P_{-P_l}) - f_j^2(N_{-P_k}, v_{-P_k}, L_{-P_k}, P_{-P_k}) = f_i^2(N, v, L, P) - f_j^2(N, v, L, P).$$

Hence, $f_i^1(N, v, L, P) - f_i^2(N, v, L, P) = f_j^1(N, v, L, P) - f_j^2(N, v, L, P)$ for any $i \in P_k \in P$ and $j \in P_l \in P$, $P_k \neq P_l$. This, in turn, implies that there exists an $\alpha \in \mathbb{R}$ such that $f_i^1(N, v, L, P) - f_i^2(N, v, L, P) = \alpha$, for all $i \in N$. It follows from graph efficiency that $\sum_{i \in N} f_i^1(N, v, L, P) = \sum_{K \in C^L(N)} v(K) = \sum_{i \in N} f_i^2(N, v, L, P)$ so that

$$0 = \sum_{i \in N} \Big(f_i^1(N, v, L, P) - f_i^2(N, v, L, P) \Big) = |N|\alpha.$$

Since $|N| > 0$ this means that $\alpha = 0$, so that $f^1(N, v, L, P) = f^2(N, v, L, P)$, a contradiction. $\square$

Replacing graph efficiency by efficiency, it can be shown in a similar way as in Kamijo (2011) that we obtain a characterization of the collective value on the class of games with coalition and graph structure. Since graph efficiency takes into account that the players in N can only realize the sum of the worths of the components of the graph L, the value ϕ can be seen as a modification of Kamijo's collective value defined on the class of games with only a coalition structure to the class of games with coalition and graph structure.

Proposition 4.10 *A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ satisfies efficiency, balanced contributions and collective balanced contributions if and only if $f(N, v, L, P) = Ka(N, v, P)$ for all $i \in N$ and every $(N, v, L, P) \in \mathcal{G}_{CG}$.*

4.2 Characterization of the partition-graph value

The partition-graph value can be characterized by axioms similar to those that characterize the Myerson value for graph games. First, recall that component efficiency of Myerson (1977) for graph games states that players in one component of the graph share exactly the worth of their component. This axiom is stated in terms of games with coalition and graph structure in Vázquez-Brage *et al.* (1996) and implies that for every component K in $C^L(N)$ the players of K distribute the worth $v(K)$ among themselves.[7]

Axiom 4.11 Component efficiency
A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ is component efficient if for any $(N, v, L, P) \in \mathcal{G}_{CG}$ it holds that $\sum_{i \in K} f_i(N, v, L, P) = v(K)$, for all $K \in C^L(N)$.

The partition-graph value does not satisfy this axiom in general, because it takes into account that within a component K of L players can only realize the sum of the worths of the coalitions T in K/P. However, it satisfies that the players in every component K of L distribute the worths of the coalitions in K/P among themselves, as stated in the next axiom.

Axiom 4.12 Partition component efficiency
A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ is partition component efficient if for any $(N, v, L, P) \in \mathcal{G}_{CG}$ it holds that $\sum_{i \in K} f_i(N, v, L, P) = \sum_{T \in K/P} v(T)$, for all $K \in C^L(N)$.

When L is connected, then N is the unique component of L and also the unique element of N/P. In this case partition component efficiency implies efficiency.

Next, Myerson (1977)'s fairness for TU-games with graph structure is translated to the setting of TU-games with coalition and graph structure. We require that, given two players i and j that are linked in the graph L (i.e. $\{i, j\} \in L$), both their values change by the same amount when the link between them is severed. To simplify notation we write $L \setminus \{i, j\}$ instead of $L \setminus \{\{i, j\}\}$.

Axiom 4.13 Fairness
A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ is fair if for any $(N, v, L, P) \in \mathcal{G}_{CG}$ it holds that

$$f_i(N, v, L, P) - f_i(N, v, L\setminus\{i, j\}, P) = f_j(N, v, L, P) - f_j(N, v, L\setminus\{i, j\}, P)$$

for all $i, j \in N$ such that $\{i, j\} \in L$.

The axioms 4.12 and 4.13 characterize the partition-graph value.

Theorem 4.14 *A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ satisfies partition component efficiency and fairness if and only if $f(N, v, L, P) = \psi(N, v, L, P)$ for every $(N, v, L, P) \in \mathcal{G}_{CG}$.*

[7]Note that any solution for TU-games with graph and coalition structure that satisfies component efficiency, also satisfies graph efficiency.

Proof. First, we show that ψ satisfies partition component efficiency and fairness. For every $K \in C^L(N)$ we have

$$\sum_{i\in K} \psi_i(N,v,L,P) = \sum_{i\in K} Sh_i(N,(v|_P)^L) = \sum_{i\in K} \sum_{\{S\subseteq N | i\in S\}} \frac{\Delta_{(v|_P)^L}(S)}{|S|} =$$

$$\sum_{S\subseteq K} \Delta_{(v|_P)^L}(S) = (v|_P)^L(K) = \sum_{S\in C^L(K)} v|_P(S) = v|_P(K) = \sum_{T\in K/P} v(T),$$

where the first two equalities follow by definition, the third by rearranging terms, the fourth by the expression for the dividends and the last three again by definition. Hence, ψ satisfies partition component efficiency.

Next, for every pair $i,j \in N$ such that $\{i,j\} \in L$ we have

$$\psi_i(N,v,L,P) - \psi_i(N,v,L\backslash\{i,j\},P) = My_i(N,v|_P,L) - My_i(N,v|_P,L\backslash\{i,j\}) =$$

$$My_j(N,v|_P,L) - My_j(N,v|_P,L\backslash\{i,j\}) = \psi_j(N,v,L,P) - \psi_j(N,v,L\backslash\{i,j\},P),$$

where the first and the last equality follow by definition and the second because the value of Myerson (1977) satisfies fairness for TU-games with graph structure (see Myerson (1977)). So, ψ satisfies fairness.

Second, we show that there can be at most one value that satisfies partition component efficiency and fairness. This proceeds along the same lines as the first part of the proof of the Theorem in Myerson (1977). If $i \in K \in C^L(N)$ with $|K| = 1$, then partition component efficiency determines that $f_i(N,v,L,P) = \psi_i(N,v,L,P)$.

Next, suppose that there are two different values f^1 and f^2 that both satisfy partition component efficiency and fairness. Let L be a graph with a minimum number of links (elements of L) such that $f^1(N,v,L,P) \neq f^2(N,v,L,P)$ (note that $|L| > 0$ in this case.) If $\{i,j\} \in L$ is a given link of L then it follows by the minimality of L that $f^1(N,v,L\backslash\{i,j\},P) = f^2(N,v,L\backslash\{i,j\},P)$. By the fairness axiom (and rearranging terms) we therefore have that

$$f_i^1(N,v,L,P) - f_j^1(N,v,L,P) = f_i^1(N,v,L\backslash\{i,j\},P) - f_j^1(N,v,L\backslash\{i,j\},P) =$$

$$f_i^2(N,v,L\backslash\{i,j\},P) - f_j^2(N,v,L\backslash\{i,j\},P) = f_i^2(N,v,L,P) - f_j^2(N,v,L,P).$$

Since this holds for any $\{i,j\} \in L$ it also holds for all $i,j \in K \in C^L(N)$. Hence there exists an $\alpha_K(L) \in \mathbb{R}$ such that

$$f_i^1(N,v,L,P) - f_i^2(N,v,L,P) = \alpha_K(L)$$

for all $i \in K \in C^L(N)$. Note that $\alpha_K(L)$ depends only on K and L but not on i. It follows from partition component efficiency that $\sum_{i\in K} f_i^1(N,v,L,P) = \sum_{T\in K/P} v(T) =$

$\sum_{i \in K} f_i^2(N, v, L, P)$ so that

$$0 = \sum_{i \in K} \Big(f_i^1(N, v, L, P) - f_i^2(N, v, L, P) \Big) = |K| \alpha_K(L).$$

Since $|K| > 1$ this implies that $\alpha_K(L) = 0$, so that $f^1(N, v, L, P) = f^2(N, v, L, P)$, a contradiction. □

Replacing partition component efficiency by component efficiency, it can be shown in a similar way as in Myerson (1977) that we obtain a characterization of the Myerson value on the class of games with coalition and graph structure. Since partition component efficiency takes into account that the players of a component K of L can only realize the sum of the worths of the elements of K/P, the value ψ can be seen as a modification of the Myerson value defined on the class of games with only a graph structure to the class of games with coalition and graph structure.

Proposition 4.15 *A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ satisfies component efficiency and fairness if and only if $f(N, v, L, P) = My(N, v, L)$ for every $(N, v, L, P) \in \mathcal{G}_{CG}$.*

5 An economic example

In this section we give an economic example of TU-games with coalition and graph structure.[8] Consider an international network of natural gas pipelines. Such a network can be represented by a graph, where the nodes in the graph correspond to cities and the links in the graph to pipelines between the cities. The international aspect of the network can be captured by the coalition structure where cities that are located in the same country are elements of the same union in the coalition structure.

As an illustration, suppose that there are five cities $N = \{1, 2, 3, 4, 5\}$. City 1 supplies natural gas and cities 2, 4 and 5 have demand for it. The cities are only able to trade the natural gas through the negotiation of (binding) bi- or multilateral contracts so that we can represent this situation by a TU-game. When city 1 sells its natural gas to city 2 this creates a surplus of 3, but when it sells it to city 4 or 5 it creates a surplus of 10. In terms of a TU-game this would mean that $v(\{1, 2\}) = 3$, $v(\{1, 4\}) = 10$, $v(\{1, 5\}) = 10$. When city 3 would join the coalition of city 1 and 2, city 1 and 4 or city 1 and 5 this would have no effect on the worth of the coalition because city 3 has no demand for natural gas.

[8] Political applications are given in Vázquez-Brage *et al.* (1996) and in Alonso-Meijide *et al.* (2009). They applied their values to situations of political power in parliaments with coalition and graph structured relations between its members.

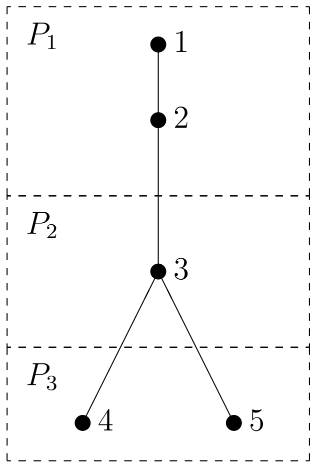

Figure 2: International network of natural gas pipelines.

Hence, $v(\{1,2,3\}) = 3$, $v(\{1,3,4\}) = 10$ and $v(\{1,3,5\}) = 10$. However, when city 1, 2 and 4, city 1,2 and 5 or city 1,4 and 5 decide to cooperate, city 1 supplies some natural gas to both of the cities that have demand for it so that $v(\{1,2,4\}) = 11$, $v(\{1,2,5\}) = 11$ and $v(\{1,4,5\}) = 11$. Also now, when city 3 joins this has no effect: $v(\{1,2,3,4\}) = 11$, $v(\{1,2,3,5\}) = 11$, $v(\{1,3,4,5\}) = 11$. Finally, when city 1 cooperates with cities 2, 4 and 5 or with all cities this creates a surplus of 12, $v(\{1,2,4,5\}) = 12$, $v(\{1,2,3,4,5\}) = 12$. When we let $v(S)$ be zero for any other coalition S we obtain a TU-game.

In this TU-game city 1 is able to supply natural gas to the other cities without any restrictions. In reality, though, the delivery of natural gas to a city requires a system of pipelines. Suppose that there is such a system of pipelines between the cities and that this system can be represented by the graph $L = \{\{1,2\},\{2,3\},\{3,4\},\{3,5\}\}$. This network of natural gas pipelines is displayed in Figure 2. What is interesting about this network is that, although city 3 has no demand for natural gas, it is in between city 1, 4 and 5 so that it is necessary in establishing cooperation between any of the coalitions involving at least two of these cities.

As a last step we introduce the countries in which the five cities are located. Cities 1 and 2 are located in country 1, $P_1 = \{1,2\}$, city 3 is located in country 2, $P_2 = \{3\}$ and cities 4 and 5 are located in country 3, $P_3 = \{4,5\}$, so that $P = \{\{1,2\},\{3\},\{4,5\}\}$. This coalition structure implies that cities can cooperate within their country and that countries can cooperate (and when they do so force all their constituent cities to cooperate). For example, city 1 is able to cooperate with city 2 because they are located in the same country, but city 1, 2, 3 and 4 are not able to cooperate in a coalition because this would require the approval of city 5 (which is in the same country as city 4).

We now have a TU-game with coalition and graph structure (N, v, L, P) that represents the cooperation possibilities of the cities that are able to trade natural gas using an international network of pipelines. A question that we can now ask ourselves is the following: what is the payoff that a city can expect facing the situation described above. Or, in the words of Shapley: what is the a priori assessment of the utility of becoming involved in the TU-game with coalition and graph structure (N, v, L, P) for each city? Different values provide different answers to this question, as can be seen in Table 5.1.

Player	Sh	My	Ka	VGC	ϕ	ψ
1	$7\frac{2}{3}$	$4\frac{1}{10}$	$3\frac{3}{4}$	3	$3\frac{3}{10}$	$3\frac{3}{10}$
2	$\frac{2}{3}$	$4\frac{1}{10}$	$3\frac{3}{4}$	3	$3\frac{3}{10}$	$3\frac{3}{10}$
3	0	$2\frac{3}{5}$	0	3	$1\frac{4}{5}$	$1\frac{4}{5}$
4	$1\frac{5}{6}$	$\frac{3}{5}$	$2\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{4}{5}$	$1\frac{4}{5}$
5	$1\frac{5}{6}$	$\frac{3}{5}$	$2\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{4}{5}$	$1\frac{4}{5}$

Table 5.1 Comparison between various values.

In Table 5.1, Sh represents the value that assigns to every TU-game with coalition and graph structure the Shapley value of the unrestricted game, i.e. $Sh_i(N, v, L, P) = Sh_i(N, v)$ for all $i \in N$. Similarly, $My(N, v, L, P) = My(N, v, L)$ and $Ka(N, v, L, P) = Ka(N, v, P)$. The VGC value is the value for TU-games with coalition and graph structure proposed in Vázquez-Brage *et al.* (1996) and the values ϕ and ψ are the graph-partition value, respectively the partition-graph value.[9]

What is interesting, but not surprising, about these values is that the Shapley value and the collective value assign a value of zero to city 3. The reason for this is that these values do not take into account the role that city 3 plays in the network of natural gas pipelines. The Myerson value does take this into account, but does not consider the fact that the cities are located in several countries. This explains why the Myerson value gives a relatively low value to city 4 and 5, that are located in the same country. Further, it seems that the VCG value overestimates the role of city 3. The two new values ϕ and ψ, which happen to be equivalent in this example, are therefore, in our opinion, the most reasonable payoff expectations for the cities in this example.

[9]Since the values of Alonso-Meijide *et al.* (2009) are not efficient, these values are not included in the table.

6 Concluding remarks

In this paper we introduced two new values for TU-games with coalition and graph structure. The first is the Shapley value of the graph-partition restricted TU-game, a game in which the dividend of any coalition that is neither a subset of a union nor a union of unions is zero. The second is the Shapley value of the partition-graph restricted TU-game, a game in which the dividend of any coalition that is not connected in the graph is zero. We showed that the Shapley value of the graph-partition restricted TU-game can be characterized by the axioms of graph efficiency, balanced contributions and collective balanced contributions, and that the Shapley value of the partition-graph restricted TU-game can be characterized by the axioms of partition component efficiency and fairness. Finally, we applied our values to an economic example of cities, located in several countries, trading natural gas through a network of pipelines. In future research we plan to investigate when, and how, the two values that we introduced in this paper differ and which value is the more appropriate one to apply in specific situations.

Finally, we would like to remark that the axioms characterizing the two new values in Theorem 4.9, respectively 4.14 are logically independent. In addition, the partition-graph value ψ also satisfies balanced contributions, which follows immediately from the fact that this property is also satisfied by the Myerson value. In contrast, the graph-partition value ϕ does not satisfy fairness. However, we could separate the fairness axiom by distinguishing the deletion of links between players in the same union from the deletion of links between players from different unions.

Axiom 6.1 Internal fairness
A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ satisfies internal fairness if for any $(N, v, L, P) \in \mathcal{G}_{CG}$ it holds that

$$f_i(N, v, L, P) - f_i(N, v, L \backslash \{i, j\}, P) = f_j(N, v, L, P) - f_j(N, v, L \backslash \{i, j\}, P)$$

for all $i, j \in N$ such that $\{i, j\} \in L$ and $\{i, j\} \subseteq P_k$ for some $k \in M$.

Axiom 6.2 External fairness
A value f on the class of TU-games with coalition and graph structure $\mathcal{G}_{CG}$ satisfies external fairness if for any $(N, v, L, P) \in \mathcal{G}_{CG}$ it holds that

$$f_i(N, v, L, P) - f_i(N, v, L \backslash \{i, j\}, P) = f_j(N, v, L, P) - f_j(N, v, L \backslash \{i, j\}, P)$$

for all $i, j \in N$ such that $\{i, j\} \in L$, $i \in P_k$, $j \in P_q$ for $k, q \in M$, $k \neq q$.

Now, it can be shown that the graph-partition value ϕ satisfies internal fairness, but does not satisfy external fairness. In fact, 'combining' the uniqueness parts of the proofs of Theorems 4.9 and 4.14, it can be shown that there is at most one value satisfying graph efficiency, internal fairness and collective balanced contributions, showing that these axioms characterize the graph-partition value ϕ. Comparing this last axiomatization of the graph-partition value ϕ with the given axiomatization of the partition-graph value ψ, we see that both are characterized by an internal axiom (they both satisfy internal fairness as well as balanced contributions), an external axiom (collective balanced contributions, respectively, external fairness) and an efficiency axiom (graph efficiency, respectively, partition component efficiency). We summarize the properties in Table 6.3.

	My	Ka	ϕ	ψ
efficiency		+		
graph efficiency	+		+	
component efficiency	+			
partition component efficiency				+
balanced contributions	+	+	+	+
collective balanced contributions		+	+	
fairness	+	+		+
internal fairness	+	+	+	+
external fairness	+	+		+

Table 6.3 Table of axioms satisfied by the values.

References

Alonso-Meijide, J.M., M. Álvarez-Mozos and M.G. Fiestras-Janeiro (2009), Values of games with graph restricted communication and a priori unions, *Mathematical Social Sciences* 58, 202-213.

Aumann, R.J. and J.H. Drèze (1974), Cooperative games with coalition structures, *International Journal of Game Theory* 3, 217-237.

Banzhaf, J.F. (1965), Weighted voting does not work: a mathematical analysis, *Rutgers Law Review* 19, 317-343.

Brink, J.R. van den, and G. van der Laan (2005), A class of consistent share functions for cooperative games in coalition structure, *Games and Economic Behavior* 51, 193-212.

Harsanyi, J.C. (1963), A simplified bargaining model for the n-person cooperative game, *International Economic Review* 4, 194-220.

Kamijo, Y. (2011), The collective value: a new solution for games with coalition structures, *TOP*, doi:10.1007/s11750-011-0191-y.

Myerson, R.B. (1977), Graphs and cooperation in games, *Mathematics of Operations Research* 2, 225-229.

Myerson, R.B. (1980), Conference structures and fair allocation rules, *International Journal of Game Theory* 9, 169-182.

Neumann, J. von, and O. Morgenstern (1944), *Theory of Games and Economic Behavior*, Princeton University Press, Princeton.

Owen, G. (1977), Values of games with a priori unions, in *Mathematical Economics and Game Theory*, ed. R. Henn and O. Moeschlin, Springer-Verlag, Berlin, 76-88.

Owen, G. (1986), Values of graph-restricted games, *SIAM Journal on Algebraic and Discrete Methods* 7, 210-220.

Shapley, L.S. (1953), A value for n-person games, in *Contributions to the Theory of Games, Vol. II*, ed. H.W. Kuhn and A.W. Tucker, Princeton University Press, Princeton, 307-317.

Vázquez-Brage, M., I. García-Jurado and F. Carreras (1996), The Owen value applied to games with graph-restricted communication, *Games and Economic Behavior* 12, 42-53.